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A character formula for atypical critical $\mathfrak{gl}(m|n)$ representations labelled by composite partitions

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Abstract
Let $\mathfrak{g}$ be the Lie superalgebra $\mathfrak{gl}(m|n)$. We show how to associate a $\mathfrak{gl}(m|n)$ weight Λ to a composite partition $\overline{\nu};\mu$ with composite Young diagram $F(\overline{\nu};\mu)$. Based upon the definition of critical representations, the notion of 'critical composite partition' is introduced. It is shown that for critical composite partitions (subject to a technical restriction) the corresponding $\mathfrak{gl}(m|n)$ representation V_Λ is tame, so its character formula can be computed. This character is shown to coincide with the composite S-function $s_{\overline{\nu};\mu}(x/y)$.

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1. Introduction

Lie superalgebras and their representations continue to play an important role in the understanding and exploitation of supersymmetry in physical systems. The Lie superalgebras under consideration here, namely $\mathfrak{gl}(m|n)$ or $\mathfrak{sl}(m|n)$ (sometimes denoted by $U(m|n)$ or $SU(m|n)$), have applications in quantum mechanics [1, 2], nuclear physics [3–5], string theory [6, 7], conformal field theory [8], supergravity [9, 10], M-theory [11], lattice QCD [12–14], solvable lattice models [15], spin systems [16] and quantum systems [17]. Also their affine extensions [8, 16] or q-deformations [1, 17] play an important role. In most of the applications, these are the irreducible representations or 'multiplets' of $\mathfrak{gl}(m|n)$ that play a role.

This paper presents some new results for irreducible representations of the Lie superalgebra $\mathfrak{gl}(m|n)$ (sometimes referred to as simple $\mathfrak{gl}(m|n)$ modules). Representation theory of Lie superalgebras, and in particular of $\mathfrak{gl}(m|n)$ or its simple counterpart $\mathfrak{sl}(m|n)$, is not a straightforward copy of the corresponding theory for simple Lie algebras. The development of $\mathfrak{gl}(m|n)$ representation theory is quite remarkable. Shortly after the classification of finite-dimensional simple Lie superalgebras [18, 19], Kac considered the problem of classifying all finite-dimensional irreducible representations (irreps) of the basic classical Lie superalgebras [20]. For a subclass of these irreps, known as 'typical' representations, Kac derived a character formula closely analogous to the Weyl character formula for irreps of simple Lie algebras [20]. The problem of obtaining a character formula for the remaining 'atypical' irreps has been the subject of intensive investigation, both in the mathematics and physics literature. In the early days of Lie superalgebra representation theory, the notion of graded tensors was introduced [21], and it was believed [22, 23] that the standard methods of covariant, contravariant and mixed tensor representations with the corresponding Young techniques yield the characters of $\mathfrak{gl}(m|n)$ irreps in terms of supersymmetric S-functions. Although this is certainly true for the covariant and contravariant tensor representations [21, 24], it is not so for the mixed tensor representations, as already observed in [25, 26]. The problem is well described and analysed in [27], where furthermore a character formula for atypical $\mathfrak{gl}(m|n)$ irreps is conjectured. Since then, some partial solutions to this problem were given, e.g., for so-called generic representations [28], for singly atypical representations [29–31], or for tame representations [32]. More recently, the character problem for $\mathfrak{gl}(m|n)$ was principally solved by Serganova [33] who gave an algorithm to compute composition factor multiplicities of so-called Kac-modules, and thus indirectly the character. In [34], a substantially simpler method was conjectured to compute these composition factor multiplicities; this conjecture was proved by Brundan [35]. Still, the method using composition factor multiplicities of Kac-modules remains a rather indirect way of computing characters. Recently, there was a further breakthrough for this problem. Developing on the work of Brundan, Su and Zhang [36] managed to compute the generalized Kazhdan–Lusztig polynomials of $\mathfrak{gl}(m|n)$ irreps, leading to a relatively explicit character formula for all these irreps, and thus proving that the character formula conjectured in [27] holds.

From a computational and practical point of view, it is useful to identify characters with supersymmetric S-functions, since it is easy to work with S-functions, for which many properties are known (see the appendix). As just mentioned, this identification holds for covariant and contravariant irreps [21, 24], where the corresponding S-function is labelled by a single partition λ, but fails for mixed tensor irreps, where the corresponding S-function is labelled by a composite partition $\overline{\nu}; \mu$. In the present paper, we show that there is still another family of atypical representations for which the character is given by a (composite) S-function, namely the so-called critical $\mathfrak{gl}(m|n)$ representations.

The notion of a critical atypical representation was already introduced in [37]. It is expressing how the 'atypical roots' with respect to the highest weight of the representation are related to each other. In [37], the highest weight of the irrep is labelled by its Dynkin labels, or by its components in a standard basis. In order to make a connection with S-functions, we shall describe the highest weight here by means of a composite partition $\overline{\nu}; \mu$. In terms of this labelling, there is a combinatorial way of characterizing critical atypical representations.

Next, we use essentially the method of [38] to show (under the technical restriction of 'no overlap') that these critical atypical representations are 'tame', in the sense of Kac and Wakimoto [32]. Using their results, we construct an explicit character formula for these irreps, and we show how this formula can be rewritten in a determinantal form. Using this determinantal form, it can be shown that the character coincides with a supersymmetric composite S-function. For this last step, the technical details of the proof are given in an appendix.

We end this section by fixing some notation for the Lie superalgebra $\mathfrak{gl}(m|n)$.

Let $\mathfrak{g}$ be the Lie superalgebra $\mathfrak{gl}(m|n)$. The general linear Lie superalgebra is one of the standard families of classical Lie superalgebras. Lie superalgebras are characterized by a $\mathbb{Z}_2$-grading $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus g_{\bar{1}}$. For the general theory on classical Lie superalgebras and their representations, we refer to [18–20].

Let $\mathfrak{h} \subset \mathfrak{g}$ be the Cartan subalgebra of $\mathfrak{g}$, and $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{+1}$ be the $\mathbb{Z}$-grading that is consistent with the $\mathbb{Z}_2$-grading of $\mathfrak{g}$. Note that $\mathfrak{g}_0 = \mathfrak{g}_{\bar{0}} = \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$. The dual space $\mathfrak{h}^*$ of $\mathfrak{h}$ has a natural basis $\{\epsilon_1, \ldots, \epsilon_m, \delta_1, \ldots, \delta_n\}$, and the roots of $\mathfrak{g}$ can be expressed in terms of this basis. In the so-called *distinguished choice* [18] for a triangular decomposition of $\mathfrak{g}$, the simple root system is given by

$$\Pi = \{\epsilon_1 - \epsilon_2, \ldots, \epsilon_{m-1} - \epsilon_m, \epsilon_m - \delta_1, \delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_n\}. \tag{1.1}$$

In that case, the positive even roots are given by $\Delta_{0,+} = \{\epsilon_i - \epsilon_j | 1 \leqslant i < j \leqslant m\} \cup \{\delta_i - \delta_j | 1 \leqslant i < j \leqslant n\}$, and the positive odd roots by $\Delta_{1,+} = \{\epsilon_i - \delta_j | 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\}$.

In the distinguished basis there is only one simple root which is odd. As usual, we put

$$\rho_0 = \frac{1}{2}\left(\sum_{\alpha \in \Delta_{0,+}} \alpha\right), \qquad \rho_1 = \frac{1}{2}\left(\sum_{\alpha \in \Delta_{1,+}} \alpha\right), \qquad \rho = \rho_0 - \rho_1. \tag{1.2}$$

There is a symmetric form $(\ ,\ )$ on $\mathfrak{h}^*$ induced by the invariant symmetric form on $\mathfrak{g}$, and in the natural basis it takes the values $(\epsilon_i, \epsilon_j) = \delta_{ij}$, $(\epsilon_i, \delta_j) = 0$ and $(\delta_i, \delta_j) = -\delta_{ij}$. The odd roots are isotropic: $(\alpha, \alpha) = 0$ if $\alpha \in \Delta_1$.

The Weyl group of $\mathfrak{g}$ is the Weyl group W of $\mathfrak{g}_0$, hence it is the direct product of symmetric groups $S_m \times S_n$. For $w \in W$, we denote by $\varepsilon(w)$ its signature.

Let V be a finite-dimensional irreducible representation of $\mathfrak{g}$. Such representations are $\mathfrak{h}$-diagonalizable with weight decomposition $V = \oplus_\mu V(\mu)$, and the character is defined to be $\operatorname{ch} V = \sum_\mu \dim V(\mu)\, \mathrm{e}^\mu$, where e^μ $(\mu \in \mathfrak{h}^*)$ is the formal exponential. The irreps of $\mathfrak{g}$ are characterized by their highest weight Λ, and denoted by V_Λ. There is a one-to-one correspondence [20] between finite-dimensional irreps and highest weights $\Lambda = \sum_{i=1}^m a_i \epsilon_i + \sum_{j=1}^n b_j \delta_j$ with all $a_i - a_{i+1}$ and $b_j - b_{j+1}$ nonnegative integers. Here, it is sufficient to consider integral highest weights Λ; that is, all a_i and b_j are integers and all $a_i - a_{i+1}$ and $b_j - b_{j+1}$ nonnegative integers.

Finally, we shall use in this paper the classical notation of partitions and symmetric functions [46], such as λ' for the conjugate of a partition λ, $\ell(\lambda)$ for its length, $F(\lambda)$ for its Young diagram, etc.

2. Composite Young diagrams and composite partitions

The composite Young diagram $F(\overline{\nu}; \mu) = F(\ldots, -\nu_2, -\nu_1; \mu_1, \mu_2, \ldots)$, specified by the pair of partitions $\mu = (\mu_1, \mu_2, \ldots)$ and $\nu = (\nu_1, \nu_2, \ldots)$, consists of two conventional Young diagrams $F(\mu)$ and $F(\nu)$. The former is composed of boxes arranged in left-adjusted rows of lengths $\mu_1, \mu_2, \ldots$ (from top to bottom), and the latter of boxes arranged in right-adjusted rows of lengths $\nu_1, \nu_2, \ldots$ (from bottom to top). A manner of juxtaposition of $F(\mu)$ and $F(\nu)$ to form $F(\overline{\nu}; \mu)$ was given in [39]. To some extent this is a refining of the back-to-back notation of [40] and [41]. By way of illustration, for $\overline{\nu}; \mu = (\overline{3}, \overline{8}); (5, 3, 1)$ the composite Young diagram is displayed in (2.1)(a). Note that in $(\overline{3}, \overline{8})$ we have used the convention of putting the minus signs on top of the integers; so in this example $\mu = (5, 3, 1)$ and $\nu = (8, 3)$. We shall refer to $\overline{\nu}; \mu$ as being a 'composite partition'.

Let m and n be fixed. In the process of associating a weight of $\mathfrak{gl}(m|n)$ to a composite partition $\overline{\nu}; \mu$, there is another way to visualize $\overline{\nu}; \mu$ by putting them together in a $(m \times n)$-rectangle. The partition μ is now composed of boxes arranged in left-adjusted rows of lengths $\mu_1, \mu_2, \ldots$ starting at the top left-hand corner of this rectangle, and the partition ν of boxes

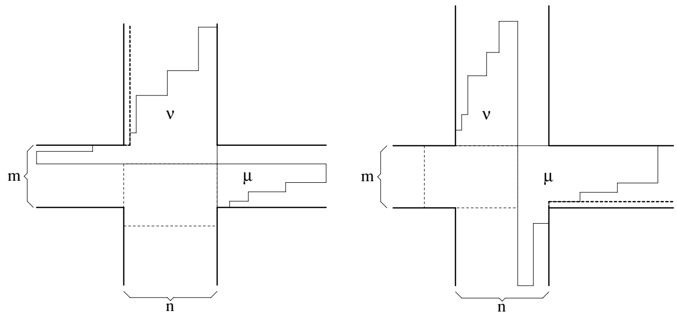
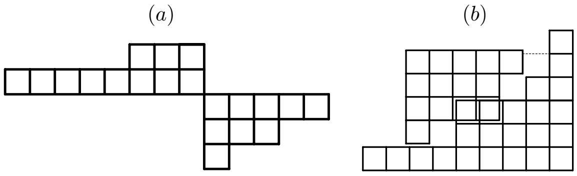


Figure 1. $F(\overline{\nu};\mu)$ with $\overline{\nu};\mu$ standard.

arranged in right-adjusted rows of lengths $\nu_1, \nu_2, \ldots$ starting at the bottom right-hand corner of the rectangle. For $\overline{\nu};\mu = (\overline{1}, \overline{1}, \overline{2}, \overline{5}, \overline{5}, \overline{9}); (5, 4, 4, 1)$ and $(m|n) = (5|7)$ this is illustrated in (2.1)(b). Observe that in this second visualization, there can be overlap between the two diagrams.

(a) (b)



(2.1)

When $\nu = 0$, the (ordinary) partition μ labels a covariant representation of $\mathfrak{gl}(m|n)$ if $\mu_{m+1} \leqslant n$; and when $\mu = 0$, $\overline{\nu}$ labels (under similar conditions) a contravariant representation of $\mathfrak{gl}(m|n)$ [24]. In both cases, the partition determines a certain highest weight Λ of the corresponding irreducible representation (or simple module) V_Λ. In [27], it was shown how to determine the highest weight Λ for the given partition μ or $\overline{\nu}$. Following this, we associate with any given composite partition $\overline{\nu};\mu$ a certain $\mathfrak{gl}(m|n)$ weight $\Lambda_{\overline{\nu};\mu}$ as follows, in the standard ϵ-δ-basis:

$$\Lambda_{\overline{\nu};\mu} = \sum_{i=1}^{m} (\mu_i - \langle \nu_{m-i+1} - n\rangle)\epsilon_i + \sum_{j=1}^{n} (\langle \mu'_j - m\rangle - \nu'_{n-j+1})\delta_j \tag{2.2}$$

where $\langle a\rangle = \max(0, a)$. Conversely, we wish to associate with any given (integral) weight Λ, in an unique way, a composite partition $\overline{\nu};\mu$. In order to make this unique, it is necessary to consider 'standard' composite partitions $\overline{\nu};\mu$.

Definition 2.1. *A composite partition $\overline{\nu};\mu$ is said to be standard if and only if one of the following conditions is fulfilled*

1. $\nu = 0$ *and* $\mu_{m+1} \leqslant n$ *(i.e., labelling a covariant representation);*
2. $\mu = 0$ *and* $\nu'_{n+1} \leqslant m$ *(i.e., labelling a contravariant representation);*
3. $\mu_m = 0$ *and* $\mu'_1 + \nu'_n \leqslant m$*;*
4. $\nu'_n = 0$ *and* $\mu_m + \nu_1 \leqslant n$*.*

These conditions are stronger than those given in [39 section 3] or [42], but necessary for the uniqueness. The possibilities 3 and 4 are illustrated in figure 1.

In order to see that the conditions $\mu'_1 + \nu'_n \leqslant m$ and $\mu_m + \nu_1 \leqslant n$ are needed for a unique correspondence, consider the weight $\Lambda = (5, 2, 1; 1, \overline{2}, \overline{3}, \overline{4})$ in $\mathfrak{gl}(3|4)$. All three of the composite partitions $\overline{\nu};\mu = (\overline{1},\overline{2},\overline{3},\overline{3}); (5,2,1,1)$, $\overline{\nu};\mu = (\overline{1},\overline{2},\overline{3},\overline{4}); (5,2,1,1,1)$ or $\overline{\nu};\mu = (\overline{1},\overline{3},\overline{4},\overline{5}); (5,2,2,2,1,1)$ would lead, by (2.2), to Λ as the corresponding weight. However, only the first satisfies the condition $\nu'_n = 0$ and $\mu_m + \nu_1 \leqslant n$.

Observe that the conditions of definition 2 also impose a limitation on the possible weights. For example $\Lambda = (2,1,1;\overline{1},\overline{1},\overline{2},\overline{3})$ is a highest weight in $\mathfrak{gl}(3|4)$, but it is impossible to find a corresponding standard composite partition $\overline{\nu};\mu$. This, however, is not a real problem in the present context. Indeed, suppose that $\Lambda' = \Lambda + r\left(\sum_{i=1}^m \epsilon_i - \sum_{j=1}^n \delta_j\right)$, then the characters of the corresponding irreducible representations satisfy $\operatorname{ch} V_{\Lambda'} = \left(\prod_{i=1}^m \mathrm{e}^{\epsilon_i}\prod_{j=1}^n \mathrm{e}^{-\delta_j}\right)^r \operatorname{ch} V_\Lambda$, with $r\in\mathbb{Z}$. And for any given Λ', one can determine the corresponding Λ such that it can be linked to a standard composite partition. This means that we are working in $\mathfrak{sl}(m|n)$ rather than in $\mathfrak{gl}(m|n)$: in $\mathfrak{sl}(m|n)$ the weights Λ and Λ' coincide. For example, $\Lambda = (2,1,1;\overline{1},\overline{1},\overline{2},\overline{3})$ can be rewritten as $\Lambda = (1,0,0;0,0,\overline{1},\overline{2}) + (1,1,1;\overline{1},\overline{1},\overline{1},\overline{1})$. The standard composite partition $\overline{\nu};\mu$ corresponding to $(1,0,0;0,0,\overline{1},\overline{2})$ is now $\overline{\nu};\mu = (\overline{1},\overline{2});(1)$.

One can show that this yields a unique correspondence between integral highest weights of $\mathfrak{sl}(m|n)$ and standard composite partitions.

Given a composite partition $\overline{\nu};\mu$, one can define the corresponding S-function [22, 23]. The complete supersymmetric polynomials are $h_r(x/y) = \sum_{k=0}^r h_{r-k}(x)e_k(y)$, with h_{r-k} the complete symmetric polynomials and e_k the elementary symmetric polynomials [46]. Furthermore, let $\dot{h}_r(x/y) = h_r(\overline{x}/\overline{y})$, where $\overline{x_i} = 1/x_i$ and $\overline{y_j} = 1/y_j$. Then,

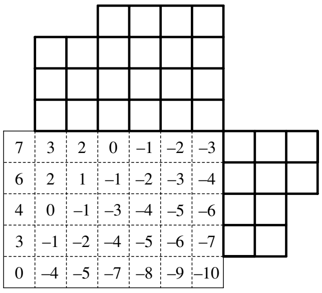
$$s_{\overline{\nu};\mu}(x/y) = \det\begin{pmatrix} \dot{h}_{\nu_l+k-l}(x/y) & h_{\mu_j-k-j+1}(x/y) \\ \dot{h}_{\nu_l-i-l+1}(x/y) & h_{\mu_j+i-j}(x/y)\end{pmatrix} \tag{2.3}$$

where i, j, k resp. l runs from top to bottom, from left to right, from bottom to top, resp. from right to left. For $\nu = 0$, this S-function is a so-called supersymmetric Schur function, and it yields the character of the (covariant) representation with highest weight $\Lambda_{0;\mu}$. Similarly, for $\mu = 0$, this yields the character of a contravariant representation. For a genuine composite partition, the functions $s_{\overline{\nu};\mu}(x/y)$ have many properties similar to ordinary Schur functions [39, 42–45] (see also the appendix to this paper). In the early days of representation theory, it was therefore believed that $s_{\overline{\nu};\mu}(x/y)$ always yields the character of a $\mathfrak{gl}(m|n)$ representation [22, 23]. This turned out to be false, making the character problem for $\mathfrak{gl}(m|n)$ a difficult one. Despite this negative answer, it is still surprising how often $s_{\overline{\nu};\mu}(x/y)$ yields the correct character of a $\mathfrak{gl}(m|n)$ irrep. So far, there are no conditions known when this is actually the case, except the rule that 'm and n should be sufficiently large compared to the number of boxes in $\overline{\nu};\mu$' [26]. In the present paper, we give a clear condition (criticality) under which $s_{\overline{\nu};\mu}(x/y)$ is actually the character of an irreducible $\mathfrak{gl}(m|n)$ representation. Note that also for typical representations, $s_{\overline{\nu};\mu}(x/y)$ yields the correct character (an unpublished result obtained by King).

3. Atypical and critical representations in $\mathfrak{gl}(m|n)$

Let $\Lambda \in \mathfrak{h}^*$; the *atypicality* of Λ, denoted by $\operatorname{atyp}(\Lambda)$, is the maximal number of linearly independent roots β_i such that $(\beta_i,\beta_j) = 0$ and $(\Lambda,\beta_i) = 0$ for all i and j [32]. Such a set $\{\beta_i\}$ is called a Λ-maximal isotropic subset of Δ. If $\operatorname{atyp}(\Lambda) = 0$, Λ is called typical.

For a simple $\mathfrak{g}$ module V_Λ with highest weight Λ, the atypicality of V_Λ is $\operatorname{atyp}(\Lambda+\rho)$. Given a composite partition $\overline{\nu};\mu$, let us consider the atypicality of $V_{\Lambda_{\overline{\nu};\mu}}$ (to be denoted by $V_{\overline{\nu};\mu}$), first in the distinguished basis. For this purpose it is sufficient to compute the numbers $(\Lambda_{\overline{\nu};\mu}+\rho, \beta_{ij})$, with $\beta_{ij} = \epsilon_i - \delta_j$, for $1\leqslant i\leqslant m$ and $1\leqslant j\leqslant n$, and count the number of zeros. It is convenient to put the numbers $(\Lambda_{\overline{\nu};\mu}+\rho,\beta_{ij})$ in a $(m\times n)$-matrix (the atypicality matrix $A(\Lambda_{\overline{\nu};\mu})$ [27, 30]), and give the matrix entries in the (m, n)-rectangle. This is illustrated for $\mathfrak{gl}(5|7)$ and $\overline{\nu}; \mu = (\overline{4}, \overline{6}, \overline{6}, \overline{6}); (3, 3, 2, 2)$ in (3.1).



(3.1)

With the notation of [37], we distinguish between *normal, critical and quasicritical* related roots of the $(\Lambda + \rho)$-isotropic set. Consider the set of odd roots $\{\gamma_1, \ldots, \gamma_a\}$ with $\gamma_s = \beta_{i_s,j_s}$ such that $\left(\Lambda_{\overline{\nu};\mu} + \rho, \beta_{i_s,j_s}\right) = 0$ where $j_1 < j_2 < \cdots < j_a$. Note that $a = \text{atyp}(\Lambda_{\overline{\nu};\mu})$ and that $\gamma_1, \ldots, \gamma_a$ are ordered from the bottom left-hand corner to the top right-hand corner. Let x_{pq} with $1 \leqslant p < q \leqslant a$ be the entry in $A(\Lambda_{\overline{\nu};\mu})$ at the intersection of the column containing the γ_p zero with the row containing the γ_q zero and x_{qp} the entry at the intersection of the row containing the γ_p zero and the column containing the γ_q zero. As shown in [37], $x_{pq} = -x_{qp}$ and therefore $A(\Lambda_{\overline{\nu};\mu})$ has the following form:

$$A(\Lambda_{\overline{\nu};\mu}) = \begin{pmatrix} & \vdots & & \vdots & & \vdots & & \vdots & \\ \ldots & x_{1t} & \ldots & x_{2t} & \ldots & & \ldots & 0 & \ldots \\ & \vdots & & \vdots & & \vdots & & \vdots & \\ \ldots & & \ldots & & \ldots & & \ldots & & \ldots \\ & \vdots & & \vdots & & \vdots & & \vdots & \\ \ldots & x_{13} & \ldots & x_{23} & \ldots & 0 & \ldots & -x_{3t} & \ldots \\ & \vdots & & \vdots & & \vdots & & \vdots & \\ \ldots & x_{12} & \ldots & 0 & \ldots & -x_{23} & \ldots & -x_{2t} & \ldots \\ & \vdots & & \vdots & & \vdots & & \vdots & \\ \ldots & 0 & \ldots & -x_{12} & \ldots & -x_{13} & \ldots & -x_{1t} & \ldots \\ & \vdots & & \vdots & & \vdots & & \vdots & \end{pmatrix} \qquad (3.2)$$

Let h_{pq} be the hook length between the zeros corresponding to γ_p and γ_q, i.e., the number of steps needed to go from the γ_p zero of $A(\Lambda_{\overline{\nu};\mu})$ via x_{pq} to the γ_q zero, where the zeros themselves are included in the count. In example (3.1), Λ is three-fold atypical with $\gamma_1 = \beta_{51}$, $\gamma_2 = \beta_{32}$ and $\gamma_3 = \beta_{14}$. The hook lengths are $h_{12} = 4$, $h_{13} = 8$ and $h_{23} = 5$.

Definition 3.1. *Let Λ be a highest weight of $\mathfrak{gl}(m|n)$ with atyp$(\Lambda) = a$ and atypical roots $\{\gamma_1, \ldots, \gamma_a\}$. Then for every $1 \leqslant p < q \leqslant a$: γ_p and γ_q are normally related if and only if $x_{pq} + 1 > h_{pq}$; γ_p and γ_q are quasicritically related if and only if $x_{pq} + 1 = h_{pq}$; γ_p and γ_q are critically related if and only if $x_{pq} + 1 < h_{pq}$.*

In example (3.1), $x_{12} + 1 = 5$, $x_{13} + 1 = 8$ and $x_{23} + 1 = 4$. Thus, γ_1 and γ_2 are normally related, γ_1 and γ_3 are quasicritically related and γ_2 and γ_3 are critically related.

If each couple (γ_i, γ_{i+1}) $(i = 1, 2, \ldots, a-1)$ is critically related, then all elements of $\{\gamma_1, \ldots, \gamma_a\}$ are critically related. Then the composite partition $\overline{\nu}; \mu$, the highest weight $\Lambda_{\overline{\nu};\mu}$ and the representation $V_{\Lambda_{\overline{\nu};\mu}} \equiv V_{\overline{\nu};\mu}$ are called *critical*. This coincides with the notion of *totally connected*, as described in [34, 36]. There is a simple combinatorial way to check criticality:

Proposition 1. *Suppose $\overline{\nu}; \mu$ is standard in $\mathfrak{gl}(m|n)$ with atyp$(\Lambda_{\overline{\nu};\mu} + \rho) = a$. Let $\gamma_s = \beta_{i_s, j_s}$ so that $(\Lambda_{\overline{\nu};\mu} + \rho, \gamma_s) = 0$ $(s = 1, \ldots, a)$ and*

$$\mathcal{M} = \left\{\mu_{i_1} + m - i_1, \mu_{i_1 - 1} + m - i_1 + 1, \ldots, \mu_{i_a} + m - i_a\right\},$$
$$\mathcal{N} = \left\{\nu'_{j_1} + n - j_1, \nu'_{j_1 - 1} + n - j_1 + 1, \ldots, \nu'_{j_a} + n - j_a\right\}.$$

Then the composite partition $\overline{\nu}; \mu$ is critical for $\mathfrak{gl}(m|n)$ if and only if

$$\mathcal{M} \cup \mathcal{N} = \left\{\mu_{i_1} + m - i_1, \mu_{i_1} + m - i_1 + 1, \ldots, \mu_{i_1} + m - i_a + j_a - j_1 - a + 1\right\},$$

i.e., if and only if $\mathcal{M} \cup \mathcal{N}$ is a set of consecutive integers.

Proof. Suppose $\mathcal{M} \cup \mathcal{N} \neq \left\{\mu_{i_1} + m - i_1, \mu_{i_1} + m - i_1 + 1, \ldots, \mu_{i_1} + m - i_a + j_a - j_1 - a + 1\right\}$. This means that at least one integer is missing between $\mu_{i_1} + m - i_1$ and $\mu_{i_a} + m - i_a$. So, there exists a p such that

$$x_{p,(p+1)} > i_p - i_{p+1} + j_{p+1} - j_p - 1 = h_{p,(p+1)} - 2 \Leftrightarrow x_{p,(p+1)} + 1 \geqslant h_{p,(p+1)},$$

a contradiction. Conversely, suppose $\mathcal{M} \cup \mathcal{N}$ is a set of consecutive numbers. Define, with $p < q$, the sets $\mathcal{M}^{(pq)}$ and $\mathcal{N}^{(pq)}$ as

$$\mathcal{M}^{(pq)} = \left\{\mu_{i_p} + m - i_p, \mu_{i_p - 1} + m - i_p + 1, \ldots, \mu_{i_q} + m - i_q\right\},$$
$$\mathcal{N}^{(pq)} = \left\{\nu'_{j_p} + n - j_p, \nu'_{j_p - 1} + n - j_p + 1, \ldots, \nu'_{j_q} + n - j_q\right\}.$$

The set $\mathcal{M}^{(pq)} \cup \mathcal{N}^{(pq)} = \left\{\mu_{i_p} + m - i_p, \mu_{i_p} + m - i_p + 1, \ldots, \mu_{i_q} + m - i_q + j_q - j_p - q + p\right\}$ is also a set of consecutive numbers. This implies that

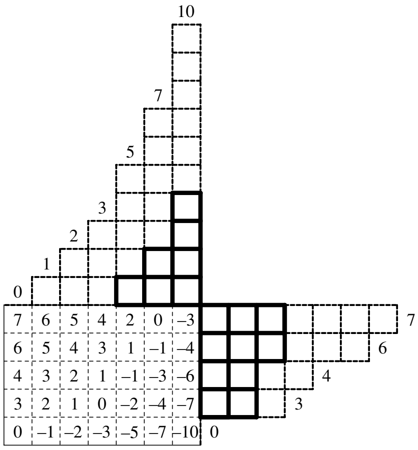
$$\mu_{i_q} + m - i_q = \mu_{i_p} + m - i_q + j_q - j_p - q + p \Leftrightarrow \mu_{i_q} - \mu_{i_p} = j_q - j_p - q + p$$

for every $p < q$. So, with $h_{pq} = j_q - j_p - i_q + i_p + 1$,

$$\begin{aligned} x_{pq} + 1 &= \mu_{i_q} + m - i_q - \left(\nu'_{j_p} + n - j_p\right) + 1 \\ &= \mu_{i_q} + m - i_q - \left(\mu_{i_p} + m - i_p\right) + 1 \\ &= h_{pq} + p - q < h_{pq}, \end{aligned}$$

meaning that γ_p and γ_q are critically related for every p, q $(p < q)$. □

This property is illustrated for $\mathfrak{gl}(5|7)$ in example (3.3) of a critical composite partition. Note how the Young diagrams, together with the $(m \times n)$-rectangle, determine the numbers attached to these diagrams; how the differences of these numbers determine the entries in the $(m \times n)$-rectangle and hence also the zeros; and how criticality can be read off from these numbers.



$$
\begin{aligned}
&\overline{\nu};\mu = (\overline{1},\overline{1},\overline{2},\overline{3}); (3,3,2,2)\\[1ex]
&(\gamma_1,\gamma_2,\gamma_3) = (\beta_{5,1},\beta_{4,4},\beta_{1,6})\\
&\mathcal{M} = \{0,3,4,6,7\}\\
&\mathcal{N} = \{0,1,2,3,5,7\}\\
&\mathcal{M}\cup\mathcal{N} = \{0,1,2,3,4,5,6,7\}\\[1ex]
&x_{12} = 3, \qquad h_{12} = 5\\
&x_{13} = 7, \qquad h_{13} = 10\\
&x_{23} = 4, \qquad h_{23} = 6
\end{aligned}
\tag{3.3}
$$

It is easy to verify that covariant or contravariant representations are always critical. The class of critical representations is however much larger. To understand this, let us concentrate again on the above example. In particular, consider $\mu = (3, 3, 2, 2)$ fixed in $\mathfrak{gl}(5|7)$, and let us determine all possible $\nu \neq 0$ such that $\overline{\nu}; \mu$ is critical. Using proposition 3, one finds, listed according to the length of ν':

- if $\ell(\nu') = 1$, all $\overline{\nu}; \mu$ are critical;
- if $\ell(\nu') = 2$, all $\overline{\nu}; \mu$ are critical as long as $\nu'_2 \notin \{1, 2\}$;
- if $\ell(\nu') = 3$, all $\overline{\nu}; \mu$ are critical as long as $\nu'_3 \notin \{2, 3\}$.

One can continue with this description for $\ell(\nu') \geqslant 4$, but it becomes slightly more intricate (some of the representations are no longer multiply atypical, and thus also not critical). In any case, this illustrates that for given m, n and μ, one can describe the corresponding critical representations $\overline{\nu}; \mu$ using proposition 3.2, and that the class of critical representations is indeed much larger than just the covariant representations (those with $\nu = 0$) and the contravariant representations (those with $\mu = 0$). In general, it illustrates that $\overline{\nu}; \mu$ is critical if the size of μ and ν is sufficiently small compared to m and n.

In this paper, we shall construct a formula for $\mathrm{ch}(\Lambda_{\overline{\nu};\mu})$ where $\overline{\nu}; \mu$ is standard, critical and such that $\overline{\nu}; \mu$ do not overlap if represented in a $(m \times n)$-rectangle (see (2.1)(b)). In what follows, we will only consider such composite partitions $\overline{\nu}; \mu$. Let $\Lambda_{\overline{\nu};\mu}$ be the highest weight corresponding to $\overline{\nu}; \mu$. We can generalize the definition of the (m, n)-index of an ordinary partition λ (cf [38]) to composite partitions $\overline{\nu}; \mu$ in $\mathfrak{gl}(m|n)$:

Definition 3.3. *For $\overline{\nu}; \mu$ a standard composite partition, the (m, n)-index of $\overline{\nu}; \mu$ is the number*

$$
\begin{aligned}
k = \min(\{i \in \{1,\ldots,m\} | \exists j \in \{1,\ldots,n\} : \mu_i + \langle \mu'_{n-j+1} - m\rangle + (m-i)\\
= \nu'_j + \langle \nu_{m-i+1} - n\rangle + (n-j)\} \cup \{m+1\})
\end{aligned}
\tag{3.4}
$$

In what follows, k will always denote this number. In the special case where $\nu = 0$, this definition coincides with the one given in [38]. When the representation is typical k will be equal to $m + 1$; otherwise k corresponds to the smallest row number in the atypicality matrix in which there occurs a zero. Thus, in the following we shall assume that $k \leqslant m$.

Recall that Δ_+ corresponds to the distinguished choice, and Π is the distinguished set of simple roots (1.1). The highest weight of $V_{\overline{\nu};\mu}$ is given by $\Lambda_{\overline{\nu};\mu}$. With respect to another set of simple roots Π' (with the corresponding ρ'), $V_{\overline{\nu};\mu}$ has a different highest weight Λ'. We shall follow the technique of simple odd reflections, described in [38]. Denote

$\Lambda^{(1)} = \Lambda_{\overline{\nu};\mu}$, $\rho^{(1)} = \rho$ and $\Pi^{(1)} = \Pi$. Now we perform a sequence of simple odd $\alpha^{(i)}$-reflections [38]; each of these reflections preserve $\Delta_{0,+}$ but may change $\Lambda^{(i)} + \rho^{(i)}$ and $\Pi^{(i)}$. Denote the sequence of reflections by

$$\Lambda^{(1)} + \rho^{(1)}, \Pi^{(1)} \quad \stackrel{\alpha^{(1)}}{\longrightarrow} \quad \Lambda^{(2)} + \rho^{(2)}, \Pi^{(2)} \quad \stackrel{\alpha^{(2)}}{\longrightarrow} \quad \cdots \quad \stackrel{\alpha^{(f)}}{\longrightarrow} \quad \Lambda' + \rho', \Pi' \tag{3.5}$$

where, at each stage, $\alpha^{(i)}$ is an odd root from $\Pi^{(i)}$. For given $\overline{\nu}; \mu$, consider the following sequence of odd roots (with positions on row m, row $m-1, \ldots,$ row k)

$$\begin{array}{ll} \text{row } m: & \beta_{m,1}, \beta_{m,2}, \ldots, \beta_{m,\min\{n,\mu_k-k+m\}} \\ \text{row } m-1: & \beta_{m-1,1}, \beta_{m-1,2}, \ldots, \beta_{m-1,\min\{n,\mu_k-k+m-1\}} \\ \vdots & \vdots \\ \text{row } k: & \beta_{k,1}, \beta_{k,2}, \ldots, \beta_{k,\mu_k} \end{array} \tag{3.6}$$

in this particular order (i.e., starting with $\beta_{m,1}$ and ending with β_{k,μ_k}). Then, we have

Lemma 3.4. *Let $\overline{\nu}; \mu$ be standard and critical in $\mathfrak{gl}(m|n)$ and suppose ν and μ do not overlap in the $(m \times n)$-rectangle. Then the sequence (3.6) is a proper sequence of simple odd reflections for $\Lambda_{\overline{\nu};\mu}$, i.e., $\alpha^{(i)}$ is a simple odd root from $\Pi^{(i)}$. At the end of the sequence, one finds*

$$\begin{aligned} \Pi' = \big\{ & \epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \ldots, \epsilon_{k-2} - \epsilon_{k-1}, \epsilon_{k-1} - \delta_1, \delta_1 - \delta_2, \delta_2 - \delta_3, \ldots, \delta_{\mu_k-1} - \delta_{\mu_k}, \\ & \delta_{\mu_k} - \epsilon_k, \epsilon_k - \delta_{\mu_k+1}, \delta_{\mu_k+1} - \epsilon_{k+1}, \epsilon_{k+1} - \delta_{\mu_k+2}, \ldots, \delta_{\mu_k+m-k} - \epsilon_m, \epsilon_m - \delta_{\mu_k+m+1-k}, \\ & \delta_{\mu_k+m+1-k} - \delta_{\mu_k+m+2-k}, \ldots, \delta_{n-1} - \delta_n \big\}. \end{aligned} \tag{3.7}$$

Furthermore,

$$\Lambda' + \rho' = \Lambda_\lambda + \rho + \sum_{i=k+1}^{k+a-1} \sum_{j=\mu_i+1}^{\mu_k-k+i} \beta_{i,j} + \sum_{i=k+a}^{m} \sum_{j=\mu_i+1}^{\max\{0,n-\nu_{m-i+1}\}} \beta_{i,j}. \tag{3.8}$$

Proof. This proof is similar to the proof of lemma 2.3 in [38]. But observe that in the first stage (i.e., the reflections with respect to odd roots of row m), $\mu_k - k + m < n$ is not necessarily true. So the sequence of odd reflections will end either with β_{m,μ_k-k+m} or with $\beta_{m,n}$. In the first case, $\Pi^{(\mu_k-k+m+1)}$ has three odd roots; in the second case, $\Pi^{(n+1)}$ contains only two odd roots. However, in both cases the set is ready to continue the reflections with respect to the elements of row $m-1$, since $\beta_{m-1,1}$ belongs to $\Pi^{(\lambda_k-k+m+1)}$ as well as to $\Pi^{(n+1)}$. Continuing with the other stages of (3.6) leads to (3.7). Remark that this sequence of simple odd reflections can always be performed, independent of whether $\overline{\nu}; \mu$ is critical or not.

Criticality does, however, play an important role in (3.8), since the changes of the atypicality matrix at each step of the sequence are governed by

$$\begin{aligned} \Lambda^{(i+1)} + \rho^{(i+1)} &= \Lambda^{(i)} + \rho^{(i)} \text{ if } (\Lambda^{(i)} + \rho^{(i)}, \alpha^{(i)}) \neq 0, \\ \Lambda^{(i+1)} + \rho^{(i+1)} &= \Lambda^{(i)} + \rho^{(i)} + \alpha^{(i)} \text{ if } (\Lambda^{(i)} + \rho^{(i)}, \alpha^{(i)}) = 0. \end{aligned} \tag{3.9}$$

Suppose that in the original situation the first zero is in the last row, so $\gamma_1 = \beta_{m,\mu_m+1}$. This assumption does not harm the generality, as $\Lambda_{\overline{\nu};\mu} + \rho$ will not change anyway until the first zero is reached in the atypicality matrix, according to (3.9). Examining the sequence of odd reflections explicitly for the elements of row m yields

$$\Lambda^{(\min\{n,\mu_k-k+m\}+1)} + \rho^{(\min\{n,\mu_k-k+m\}+1)} = \Lambda_{\overline{\nu};\mu} + \rho + \sum_{j=\mu_m+1}^{\min\{n-\nu_1,\mu_k-k+m\}} \beta_{m,j}. \tag{3.10}$$

More generally, we have that $\mu_k - k + i \leqslant n - \nu_{m-i+1}$ if $k + 1 \leqslant i \leqslant k + a - 1$, and $\mu_k - k + i > n - \nu_{m-i+1}$ if $k + a \leqslant i \leqslant m$. This explains the two different contributions in (3.8). To see that criticality is necessary, consider two consecutive roots γ_p and γ_{p+1}. For simplicity, consider again γ_1 and γ_2 with γ_1 on row m. If γ_1 and γ_2 are critically related, then there appears a zero on row $m-1$ of the atypicality matrix after finishing the first stage, and one can continue until there appears an extra zero in the row of γ_2. If they are not critically related, then no extra zero can be obtained in this row. To follow the argument, this is illustrated in the following examples, where the atypicality matrix is given in the initial situation, after finishing the first stage, and after finishing the second (and in this case final) stage:

Critically related roots:

4	3	1	0
2	1	–1	–2
0	–1	–3	–4

→

3	2	1	0
1	0	–1	–2
1	0	–1	–2

→

3	1	0	0
3	1	0	0
1	–1	–2	–2

Noncritically related roots:

5	4	2	0
2	1	–1	–3
0	–1	–3	–5

→

4	3	2	0
1	0	–1	–3
1	0	–1	–3

→

4	2	1	0
3	1	0	–1
1	–1	–2	–3

Thus, if $\overline{\nu}; \mu$ is critical, there is at least one zero at row $m - 1$ after the first stage. This zero corresponds to $\beta_{m-1,\mu_{m-1}+1}$. At the second stage, the elements of row $m-1$ play the same role as the elements of row m in the first stage, and one continues the process. Schematically, the zeros in the atypicality matrix move up along those positions corresponding to boxes that are not covered by the Young diagrams $F(\mu)$ and $F(\nu)$. Continuing with the remaining stages leads to (3.8). □

Corollary 3.5. *The critical representation $V_{\overline{\nu};\mu}$ is tame.*

Proof. Having performed the simple odd reflections (3.6), one can compute the atypicality matrix for $\Lambda' + \rho'$ using (3.8). This gives

$$(\Lambda' + \rho', \beta_{ij}) = 0 \qquad \text{for all } (i, j) \text{ with } k \leqslant i \leqslant k + a - 1, \mu_k + 1 \leqslant j \leqslant \mu_k + a. \tag{3.11}$$

Therefore, the set

$$S_{\Lambda'} = \left\{\epsilon_k - \delta_{\mu_k+1}, \epsilon_{k+1} - \delta_{\mu_k+2}, \ldots, \epsilon_m - \delta_{\mu_k+a}\right\} \tag{3.12}$$

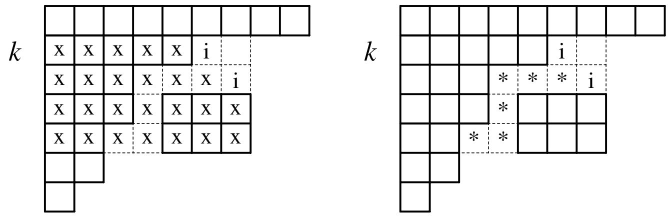
is a $(\Lambda' + \rho')$-maximal isotropic subset. Furthermore, $S_{\Lambda'} \subset \Pi'$, see (3.7). This implies that $V_{\overline{\nu};\mu}$ is tame [38]. If $\overline{\nu}; \mu$ is not critical, (3.11) does not hold, as explained in the proof of lemma 3.4. □

Let us illustrate some of these notions for $\overline{\nu}; \mu = (\overline{3}, \overline{3}); (9, 5, 3, 3, 2, 2, 1)$ in $\mathfrak{gl}(5|7)$:

(a)

	15	13	11	10	7	6	5
k	10	8	6	5	2	1	0
	7	5	3	2	−1	−2	−3
	6	4	2	1	−2	−3	−4
	4	2	0	−1	−4	−5	−6

(b) (c)



(3.13)

In (3.13)(a), the atypicality matrix associated with $\overline{\nu}; \mu$ is given. In (3.13)(b) the positions marked with 'i' refer to the $(\Lambda' + \rho')$-maximal isotropic set (3.12). For convenience, let us refer to these positions as 'the isotropic diagonal'. The positions of the odd roots that have been used for the sequence of reflections to go from $\Lambda_{\overline{\nu};\mu}$ and Π to Λ' and Π' are marked by 'x' in (3.13)(b). So, they are simply all positions to the left of the isotropic diagonal. Finally, (3.13)(c) shows the positions of those β_{ij} that appear on the right-hand side of (3.8); they are marked by '*'. These are all positions to the left of the isotropic diagonal that are not inside $F(\overline{\nu}; \mu)$. One can see from this example and others that the (m, n)-index k determines all other necessary ingredients.

4. A determinantal formula for ch($V_{\overline{\nu};\mu}$) and $s_{\overline{\nu};\mu}(x/y)$

Let $\overline{\nu}; \mu$ be a standard and critical composite partition without overlap in the $(m \times n)$-rectangle. As the $\mathfrak{g}$-module $V_{\overline{\nu};\mu}$ is tame, a character formula is known due to Kac and Wakimoto [32]. It reads, in terms of Λ':

$$\mathrm{ch}\, V_{\overline{\nu};\mu} = j_{\Lambda'^{-1}}\, \mathrm{e}^{-\rho'} R'^{-1} \sum_{w\in W} \varepsilon(w) w \left(\mathrm{e}^{\Lambda'+\rho'} \prod_{\beta\in S_{\Lambda'}} (1+\mathrm{e}^{-\beta})^{-1} \right), \tag{4.1}$$

where

$$R' = \prod_{\alpha\in\Delta_{0,+}} (1-\mathrm{e}^{-\alpha}) \Big/ \prod_{\alpha\in\Delta'_{1,+}} (1+\mathrm{e}^{-\alpha}) \tag{4.2}$$

and $j_{\Lambda'}$ is a normalization coefficient to make sure that the coefficient of $\mathrm{e}^{\Lambda'}$ on the right-hand side of (4.1) is 1. By definition of ρ and R

$$\mathrm{e}^{-\rho'} R'^{-1} = \mathrm{e}^{-\rho} R^{-1}.$$

As usual in this context we put

$$x_i = \mathrm{e}^{\epsilon_i}, \qquad y_j = \mathrm{e}^{\delta_j} \qquad (1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n). \tag{4.3}$$

Now we have

$$\mathrm{ch}\, V_{\overline{\nu};\mu} = j^{-1}_{\Lambda'_{\overline{\nu};\mu}} D^{-1} \sum_{w\in W} \varepsilon(w) w(t_{\overline{\nu};\mu}),$$

with

$$D = \frac{\prod_{1\leqslant i<j\leqslant m}(x_i - x_j) \prod_{1\leqslant i<j\leqslant n}(y_i - y_j)}{\prod_{i=1}^m \prod_{j=1}^n (x_i + y_j)} \tag{4.4}$$

and

$$t_{\overline{\nu};\mu} = \prod_{i=1}^{k-1} x_i^{\mu_i+m-i-n} \prod_{j=1}^{l-1} y_j^{\mu'_j+n-j-m} \prod_{i=k}^{k+a-1} \frac{y_{i-k+l}^r}{x_i^r(x_i+y_{i-k+l})} \prod_{i=k+a}^{n} x_i^{m-i-\nu_{m-i+1}} \prod_{j=l+a}^{n} y_j^{n-j-\nu'_{n-j+1}} \tag{4.5}$$

where $l = \mu_k + 1$ and $r = n - m + k - l$ and $j_{\Lambda'_{\overline{\nu};\mu}} = a!$ (due to symmetry there are $a!$ elements of $S_m \times S_n$ that leave $t_{\overline{\nu};\mu}$ invariant).

This expression can be written in a nicer form:

Theorem 4.1. *Let $t_{\overline{\nu};\mu}$ be given by (4.5) and $r = n - m + k - \mu_k - 1$. Then,*

$$\frac{1}{a!} \sum_{w \in S_m \times S_n} \varepsilon(w) w(t_{\overline{\nu};\mu}) = (-1)^{(m-a)(l-1)+n(m-a-k+1)} \det(C), \tag{4.6}$$

where C is the following square matrix of order $n + m - a$:

$$C = \begin{pmatrix} 0 & Y_{\mu'} & 0 \\ X_\mu & R^{(r)} & X_\nu \\ 0 & Y_{\nu'} & 0 \end{pmatrix} \qquad \text{with} \quad R^{(r)} = \left(\frac{y_j^r}{x_i^r(x_i+y_j)} \right)_{1\leqslant i\leqslant m, 1\leqslant j\leqslant n} \tag{4.7}$$

and with

$$X_\mu = \left(x_i^{\mu_j+m-n-j}\right)_{1\leqslant i\leqslant m, 1\leqslant j\leqslant k-1}, \qquad X_\nu = \left(x_i^{m-j-\nu_{m-j+1}}\right)_{1\leqslant i\leqslant m, k+a\leqslant j\leqslant m},$$
$$Y_{\mu'} = \left(y_j^{\mu'_i+n-m-i}\right)_{1\leqslant i\leqslant l-1, 1\leqslant j\leqslant n}, \qquad Y_{\nu'} = \left(y_j^{n-i-\nu'_{n-i+1}}\right)_{l+a\leqslant i\leqslant n, 1\leqslant j\leqslant n}.$$

Proof. The proof is similar to that of [38] (lemma 3.1). Apply Laplace's theorem for the expansion of $\det(C)$ with respect to columns $1, 2, \ldots, k-1, k+n, k+n+1, \ldots, n+m-a$. Keeping track of the zero blocks, one finds

$$\det(C) = (-1)^{\frac{(m-a)(m-a+1)}{2}} \sum_{1\leqslant i_1<\cdots<i_{m-a}\leqslant m} (-1)^{i_1+\cdots+i_{m-a}+(m-a)(l-1)} \det(C_x)\det(C_y), \tag{4.8}$$

where C_x is the $(m-a) \times (m-a)$-matrix consisting of rows $i_1, i_2, \ldots, i_{m-a}$ of the matrix $(X_\mu \quad X_\nu)$, and C_y is the $n \times n$-matrix

$$\begin{pmatrix} Y_{\mu'} \\ \tilde{R}^{(r)} \\ Y_{\nu'} \end{pmatrix},$$

where $\tilde{R}^{(r)}$ is obtained by removing rows $i_1, i_2, \ldots, i_{m-a}$ in $R^{(r)}$. The number of terms on the rhs of (4.8) is $\binom{m}{m-a}(m-a)!n! = m!n!/a!$; due to symmetry considerations this is the same as the number of distinct terms on the lhs of (4.6). For $(i_1, \ldots, i_{m-a}) = (1, \ldots, k-1, k+n, ..., n+m-a)$, and the diagonal term in $\det C_x$ and $\det C_y$, the contribution on the rhs of (4.8) is now easily seen to be $(-1)^{(m-a)(l-1)+n(m-a-k+1)} t_{\overline{\nu};\mu}$. But by definition of the determinant, every term on the rhs of (4.8) is (up to the overall sign factor $(-1)^{(m-a)(l-1)}$) of the form $\varepsilon(w)w(t_{\overline{\nu};\mu})$ with $w \in S_m \times S_n$. Conversely, every term of the form $\varepsilon(w)w(t_{\overline{\nu};\mu})$ appears as a term on the rhs of (4.8). It follows that (4.6) holds. □

With the same notation, one finds.

Corollary 4.2. *The character of a critical representation labelled by a standard composite partition $\overline{\nu}; \mu$ (without overlap) has the following determinantal form:*

$$\operatorname{ch} V_{\overline{\nu};\mu} = (-1)^{(m-a)(l-1)+n(m-a-k+1)} D^{-1} \det(C).$$

As an example, let $m = 4, n = 5$ and $\overline{\nu}; \mu = (\overline{1}, \overline{1}, \overline{4}); (3, 1)$. One finds

6	4	3	2	–1
3	1	0	–1	–4
1	–1	–2	–3	–6
0	–2	–3	–4	–7

$$\begin{aligned} k &= 2 \\ l &= \mu_k + 1 = 2 \\ a &= 2 \end{aligned}$$

$$\Rightarrow \quad r = n - m + k - l = 1$$
$$\Rightarrow \quad n + m - a = 7$$

Thus, according to formula (4.7),

$$\operatorname{ch} V_{(\overline{1},\overline{1},\overline{4});(3,1)} = D^{-1} \det \begin{pmatrix} 0 & y_1^2 & y_2^2 & y_3^2 & y_4^2 & y_5^2 & 0 \\ x_1 & \frac{y_1}{x_1(x_1+y_1)} & \frac{y_2}{x_1(x_1+y_2)} & \frac{y_3}{x_1(x_1+y_3)} & \frac{y_4}{x_1(x_1+y_4)} & \frac{y_5}{x_1(x_1+y_5)} & x_1^{-4} \\ x_2 & \frac{y_1}{x_2(x_2+y_1)} & \frac{y_2}{x_2(x_2+y_2)} & \frac{y_3}{x_2(x_2+y_3)} & \frac{y_4}{x_2(x_2+y_4)} & \frac{y_5}{x_2(x_2+y_5)} & x_2^{-4} \\ x_3 & \frac{y_1}{x_3(x_3+y_1)} & \frac{y_2}{x_3(x_3+y_2)} & \frac{y_3}{x_3(x_3+y_3)} & \frac{y_4}{x_3(x_3+y_4)} & \frac{y_5}{x_3(x_3+y_5)} & x_3^{-4} \\ x_4 & \frac{y_1}{x_4(x_4+y_1)} & \frac{y_2}{x_4(x_4+y_2)} & \frac{y_3}{x_4(x_4+y_3)} & \frac{y_4}{x_4(x_4+y_4)} & \frac{y_5}{x_4(x_4+y_5)} & x_4^{-4} \\ 0 & y_1^0 & y_2^0 & y_3^0 & y_4^0 & y_5^0 & 0 \\ 0 & y_1^{-3} & y_2^{-3} & y_3^{-3} & y_4^{-3} & y_5^{-3} & 0 \end{pmatrix}.$$

Thus, the determinantal formula is very explicit. The main goal of this determinantal formula however is that it allows us to make the link with another explicit formula that is even more useful, namely:

Theorem 4.3. *Let $\overline{\nu}; \mu$ be a standard and critical composite partition with no overlap. The character* $\operatorname{ch} V_{\overline{\nu};\mu}$ *is equal to* $s_{\overline{\nu};\mu}(x/y)$ *as defined in (2.3).*

The proof depends upon a double Laplace expansion of $\det(C)$, various lemmas and computational rules for ordinary and supersymmetric S-functions (labelled by ordinary or composite partitions) and an induction argument. The double Laplace expansion is similar as in the proof of [38, theorem 5.5]. The details of this proof are given in the appendix.

Appendix

We shall provide here a proof of theorem 4.3. First, we need to collect some formulae for (ordinary) S-functions labelled by a composite partition (often referred to as 'mixed' S-functions). Then we shall construct a set of formulae for the supersymmetric S-functions labelled by a composite partition. The rest of the proof follows the lines set out in section 5 of [38], however there are many technical details which need to be reinvestigated in the current case. We shall assume that the reader is familiar with notation of ordinary S-functions [46], such as $s_\lambda(x)$, $s_{\lambda/\mu}(x)$, $c^{\nu}_{\lambda\mu}$ for Littlewood–Richardson coefficients, vertical strips, etc.

The 'contravariant' S-functions are usually defined in terms of the ordinary (or 'covariant') S-functions. Suppose we have a set of variables $x = (x_1, \ldots, x_m)$. For a partition λ, let $\overline{\lambda} = (-\lambda_1, -\lambda_2, \ldots)$ and denote by $\overline{x_i} = \frac{1}{x_i}$, for all $i = 1, \ldots, m$, then,

$$s_{\overline{\lambda}}(x) = s_\lambda(\overline{x}). \tag{A.1}$$

Similarly, $s_{\overline{\lambda/\mu}}(x) = s_{\lambda/\mu}(\overline{x})$. Using the contravariant S-functions, the composite or 'mixed' S-functions are defined [42] by

$$s_{\overline{\nu};\mu}(x) = \sum_{\zeta} (-1)^{|\zeta|} s_{\overline{\nu/\zeta}}(x) s_{\mu/\zeta'}(x). \tag{A.2}$$

The product of a covariant and a contravariant S-function is given by

$$s_{\overline{\nu}}(x)s_\mu(x) = s_\nu(\overline{x})s_\mu(x) = \sum_\eta s_{\overline{\nu/\eta};\mu/\eta}(x) \tag{A.3}$$

where

$$s_{\overline{\nu/\eta};\mu/\eta}(x) = \sum_{\varphi,\psi} c^\nu_{\varphi\eta} c^\mu_{\psi\eta} s_{\overline{\varphi};\psi}(x). \tag{A.4}$$

The composite S-functions can also be written in terms of a decomposition [25] of $x = x' + x''$, namely

$$s_{\overline{\nu};\mu}(x) = \sum_{\rho,\sigma,\tau} s_{\overline{\nu/\sigma};\mu/\tau}(x') s_{\overline{\sigma/\rho};\tau/\rho}(x''). \tag{A.5}$$

In [39] the composite supersymmetric S-functions are defined in terms of ordinary composite S-functions, namely:

$$s_{\overline{\nu};\mu}(x/y) = \sum_{\eta,\zeta,\xi} s_{\overline{\nu/\xi};\mu/\zeta}(x) s_{\overline{\xi'/\eta};\zeta'/\eta}(y) = \sum_{\rho,\zeta,\xi} s_{\overline{\nu/\xi};\mu/\zeta}(x) s_{\overline{(\xi/\rho)'};(\zeta/\rho)'}(y). \tag{A.6}$$

In the same paper, the authors prove that this expression is equivalent with the definition (2.3). With formula (A.6), it is possible to prove [42, appendix 2] that (A.3) also holds in the supersymmetric case. Thus,

$$s_{\overline{\nu}}(x/y)s_\mu(x/y) = \sum_\eta s_{\overline{\nu/\eta};\mu/\eta}(x/y) \tag{A.7}$$

where

$$s_{\overline{\nu/\eta};\mu/\eta}(x/y) = \sum_{\varphi,\psi} c^\nu_{\varphi\eta} c^\mu_{\psi\eta} s_{\overline{\varphi};\psi}(x/y). \tag{A.8}$$

We need also the definition of

$$s_{\lambda/\nu\mu}(x) \equiv \sum_\varphi c^\varphi_{\nu\mu} s_{\lambda/\varphi}(x), \tag{A.9}$$

and the equality

$$\sum_\sigma c^\lambda_{\rho\sigma} c^\sigma_{\mu\nu} = \sum_\eta c^\lambda_{\mu\eta} c^\eta_{\rho\nu} = \sum_\tau c^\lambda_{\nu\tau} c^\tau_{\mu\rho}, \tag{A.10}$$

following from the three different ways in which the product $s_\mu(x)s_\nu(x)s_\rho(x)$ can be expanded.

Using (A.4) and (A.9) it is easy to see that (A.6) can be written as

$$s_{\overline{\nu};\mu}(x/y) = \sum_{\rho,\zeta,\xi} s_{\overline{\nu/\xi};\mu/\zeta}(x) s_{\overline{\xi'/\rho'};\zeta'/\rho'}(y) = \sum_{\rho,\varphi,\psi} s_{\overline{\nu/\varphi\rho};\mu/\psi\rho}(x) s_{\overline{\varphi'};\psi'}(y). \tag{A.11}$$

Formulae (A.6) and (A.11) are generalizations of (cf [46, p 90 example 23])

$$s_\lambda(x/y) = \sum_\mu s_\mu(x) s_{\lambda'/\mu'}(y) = \sum_\nu s_{\lambda/\nu}(x) s_{\nu'}(y), \tag{A.12}$$

and they can themselves be generalized to composite skew partitions:

$$s_{\overline{\nu/\eta};\lambda/\mu}(x/y) = \sum_{\rho,\sigma,\tau} s_{\overline{\nu/\eta\sigma};\lambda/\mu\tau}(x) s_{\overline{(\sigma/\rho)'};(\tau/\rho)'}(y). \tag{A.13}$$

Indeed,

$$
\begin{aligned}
s_{\overline{\nu/\eta};\lambda/\mu}(x/y) &= \sum_{\alpha,\beta} c^{\nu}_{\alpha\eta} c^{\lambda}_{\beta\mu} s_{\overline{\alpha};\beta}(x/y) && \text{(formula (A.8))}\\
&= \sum_{\rho,\sigma,\tau} \sum_{\alpha,\beta} c^{\nu}_{\alpha\eta} c^{\lambda}_{\beta\mu} s_{\overline{\alpha/\sigma};\beta/\tau}(x) s_{\overline{(\sigma/\rho)'};(\tau/\rho)'}(y) && \text{(formula(A.6))}\\
&= \sum_{\rho,\sigma,\tau} \sum_{\gamma,\delta} \left(\sum_{\alpha,\beta} c^{\nu}_{\alpha\eta} c^{\lambda}_{\beta\mu} c^{\alpha}_{\sigma\gamma} c^{\beta}_{\tau\delta} \right) s_{\overline{\gamma};\delta}(x) s_{\overline{(\sigma/\rho)'};(\tau/\rho)'}(y) && \text{(formula(A.4))}\\
&= \sum_{\rho,\sigma,\tau} \sum_{\gamma,\delta} \left(\sum_{\zeta,\xi} c^{\nu}_{\zeta\gamma} c^{\lambda}_{\xi\delta} c^{\zeta}_{\sigma\eta} c^{\xi}_{\tau\mu} \right) s_{\overline{\gamma};\delta}(x) s_{\overline{(\sigma/\rho)'};(\tau/\rho)'}(y) && \text{(formula(A.10))}\\
&= \sum_{\rho,\sigma,\tau} \sum_{\zeta,\xi} c^{\zeta}_{\sigma\eta} c^{\xi}_{\tau\mu} s_{\overline{\nu/\zeta};\lambda/\xi}(x) s_{\overline{(\sigma/\rho)'};(\tau/\rho)'}(y) && \text{(formula(A.4))}\\
&= \sum_{\rho,\sigma,\tau} s_{\overline{\nu/\eta\sigma};\lambda/\mu\tau}(x) s_{\overline{(\sigma/\rho)'};(\tau/\rho)'}(y). && \text{(formula(A.9))}
\end{aligned}
$$

Next, we introduce some further notation. Let

$$
D(x) = \prod_{1\leqslant i<j\leqslant m} (x_i - x_j) \qquad \text{and} \qquad E(x,y) = \prod_{i=1}^{m}\prod_{j=1}^{n}(x_i - y_j). \tag{A.14}
$$

Suppose $|x| = m$ and $|y| = n$. From the definition of $D(x)$ and $E(x,y)$ we derive that

$$
D(\overline{x}) = (-1)^{\frac{m(m-1)}{2}} D(x) \left(\prod_{i=1}^{m} x_i^{-m+1} \right), \tag{A.15}
$$

$$
E(\overline{x},\overline{y}) = (-1)^{mn} \left(\prod_{i=1}^{m} x_i^{-n} \right) \left(\prod_{j=1}^{n} y_j^{-m} \right) E(x,y), \tag{A.16}
$$

$$
\frac{D(\overline{x})D(\overline{y})}{E(\overline{x},\overline{y})} = (-1)^{\frac{m(m-1)}{2}+\frac{n(n-1)}{2}+mn} \left(\prod_{i=1}^{m} x_i^{n-m+1} \right) \left(\prod_{j=1}^{n} y_j^{m-n+1} \right) \frac{D(x)D(y)}{E(x,y)}. \tag{A.17}
$$

Lemma A.1. *Suppose ν is a partition, then*

$$
s_{\overline{\nu}}(x) = \frac{(-1)^{\frac{m(m-1)}{2}}}{D(x)} \left| x_j^{-\nu_i+i-1} \right| = \frac{\left| x_j^{-\nu_{m-i+1}+m-i} \right|}{D(x)}.
$$

Proof. This formula is derived from the determinantal formula for S-functions [46], applying properties of determinants:

$$
\begin{aligned}
s_{\overline{\nu}}(x) = s_{\nu}(\overline{x}) = \frac{\left| \overline{x}_j^{\nu_i+m-i} \right|}{D(\overline{x})} &= (-1)^{\frac{m(m-1)}{2}} \prod_{i=1}^{m} x_i^{m-1} \frac{\left| x_j^{-\nu_i-m+i} \right|}{D(x)}\\
&= (-1)^{\frac{m(m-1)}{2}} \frac{\left| x_j^{-\nu_i+i-1} \right|}{D(x)} = \frac{\left| x_j^{-\nu_{m-i+1}+m-i} \right|}{D(x)}.
\end{aligned}
$$

□

The following lemma will be crucial in our induction argument.

Lemma A.2. *Suppose $y = y^{(n)} = (y_1, \ldots, y_n)$. Let ν and μ be partitions, then*

$$s_{\overline{\nu};\mu}(x/y) = \sum_{a,b} s_{\overline{\nu/(1^b)};\mu/(1^a)}(x/y^{(n-1)})y_n^{a-b}.$$

Proof. We prove this statement using the formulae given earlier in this appendix.

$$\begin{aligned} s_{\overline{\nu};\mu}(x/y) &= \sum_{\varphi,\psi,\rho} s_{\overline{\nu/(\rho\varphi)};\mu/(\rho\psi)}(x)s_{\overline{\varphi'};\psi'}(y) && \text{(formula(A.11))} \\ &= \sum_{\varphi,\psi,\rho} s_{\overline{\nu/(\rho\varphi)};\mu/(\rho\psi)}(x)\left(\sum_{\eta,\kappa,\tau} s_{\overline{(\varphi/\kappa)'};(\psi/\tau)'}(y^{(n-1)})s_{\overline{\kappa/\eta'};\tau'/\eta'}(y_n)\right) && \text{(formula(A.5))} \\ &= \sum_{\kappa,\tau}\sum_{\varphi,\psi,\rho} s_{\overline{\nu/(\rho\varphi)};\mu/(\rho\psi)}(x)s_{\overline{(\varphi/\kappa)'};(\psi/\tau)'}(y^{(n-1)})\sum_{\eta} s_{\overline{\kappa'/\eta'};\tau'/\eta'}(y_n) \\ &= \sum_{\kappa,\tau} s_{\overline{\nu/\kappa};\mu/\tau}(x/y^{(n-1)})s_{\kappa'}(\overline{y_n})s_{\tau'}(y_n). && \text{(by (A.13) and (A.3))} \end{aligned}$$

As $s_\lambda(x^{(m)}) = 0$ if $\ell(\lambda) > |x^{(m)}| = m$, the right-hand side equals

$$\sum_{a,b} s_{\overline{\nu/(1^b)};\mu/(1^a)}(x/y^{(n-1)})s_{(b)}(\overline{y_n})s_{(a)}(y_n) = \sum_{a,b} s_{\overline{\nu/(1^b)};\mu/(1^a)}(x/y^{(n-1)})y_n^{a-b}.$$

□

If ν is an arbitrary t-tuple over $\mathbb{Z}$ and μ an arbitrary s-tuple over $\mathbb{Z}$, we can still define $s_{\overline{\nu};\mu}(x|y)$ through formula (2.3). Note that $\nu + \delta_t$ and $\mu + \delta_s$ must be nonnegative distinct integers for $s_{\overline{\nu};\mu}(x|y)$ to be nonzero. We need the following generalization of the previous lemma.

Lemma A.3. *Suppose $y = y^{(n)} = (y_1, \ldots, y_n)$. Let ν be an arbitrary t-tuple over $\mathbb{Z}$ and μ an arbitrary s-tuple over $\mathbb{Z}$, then*

$$s_{\overline{\nu};\mu}(x/y) = \sum_{\alpha,\beta} s_{\overline{\beta};\alpha}(x/y^{(n-1)})y_n^{a-b}, \tag{A.18}$$

where $a = |\mu-\alpha|$, $b = |\nu-\beta|$, and the sum is taken over all α and β such that $(\nu-\beta)_i \in \{0, 1\}$.

Proof. (Compare with the proof of [38, lemma 5.3].) This follows from the previous lemma and the determinant (2.3) for $s_{\overline{\nu};\mu}(x/y)$. If there are two identical columns in this determinant, then $s_{\overline{\nu};\mu}(x/y) \equiv 0$. But then also in the right-hand side of (A.18), the terms will either vanish or else cancel each other two by two. If all columns in the determinant are different, they can be permuted such that $s_{\overline{\nu};\mu}(x/y) = \pm s_{\overline{\phi};\psi}(x/y)$ where ϕ and ψ are partitions. Applying lemma A.2 and performing the inverse permutation for the S-functions in the right-hand side yields the result. □

Lemma A.4. *For $m = p + q$, let $\varphi = (\varphi_1, \ldots, \varphi_p)$ and $\sigma = (\sigma_1, \ldots, \sigma_q)$ be two partitions and $\lambda = (\varphi_1 + g - q, \ldots, \varphi_p + g - q, -\sigma_q + h, \ldots, -\sigma_1 + h)$. Suppose $g, h \in \mathbb{Z}$, then*

$$\sum_{x'+x''} \frac{\left(\prod x'\right)^g \left(\prod x''\right)^h s_\varphi(x')s_{\overline{\sigma}}(x'')}{E(x', x'')} = s_\lambda(x), \tag{A.19}$$

where the sum is over all possible decompositions $x = x' + x''$ with the size of x' equal to p and the size of x'' equal to q.

Proof. We can rewrite the left-hand side of (A.19) using the determinantal formula for S-functions and the equality

$$D(x) = (-1)^{\frac{p(p+1)}{2}+r_1+\cdots+r_p} D(x')D(x'')E(x', x''), \tag{A.20}$$

with the elements of x' denoted by $x_{r_1},\ldots,x_{r_p}$ $(r_1<\cdots<r_p)$ and those of x'' by $x_{s_1},\ldots,x_{s_q}$ $(s_1<\cdots<s_q)$:

$$\sum_{x'+x''}\frac{\left(\prod x'\right)^g\left(\prod x''\right)^h s_\varphi(x')s_{\overline{\sigma}}(x'')}{E(x',x'')}$$

$$=\sum_{x'+x''}\frac{\left(\prod x'\right)^g\left(\prod x''\right)^h}{E(x',x'')}\cdot\frac{\left|\left(x_{r_i}{}^{\varphi_j+p-j}\right)_{\substack{i=1\ldots p\\ j=1\ldots p}}\right|}{D(x')}\cdot\frac{\left|\left(x_{s_i}{}^{-\sigma_{q-j+1}+q-j}\right)_{\substack{i=1\ldots q\\ j=1\ldots q}}\right|}{D(x'')}$$

$$=\frac{1}{D(x)}\sum_{x'+x''}(-1)^{\frac{p(p+1)}{2}+r_1+\cdots+r_p}\left|\left(x_{r_i}{}^{(\varphi_j+g-q)+(p+q-j)}\right)_{\substack{i=1\ldots p\\ j=1\ldots p}}\right|\left|\left(x_{s_i}{}^{(-\sigma_{m-j+1}+h)+(m-j)}\right)_{\substack{i=1\ldots q\\ j=p+1\ldots m}}\right|.$$

The numerator of this sum is the Laplace expansion of the following determinant with respect to columns $1,\ldots,p$:

$$\left|x_i^{(\varphi_j+g-q)+(m-j)}\quad x_i^{(-\sigma_{m-j+1}+h)+(m-j)}\right|=\left|x^{\lambda+\delta_m}\right|$$

with $\lambda=(\varphi_1+g-q,\ldots,\varphi_p+g-q,-\sigma_q+h,\ldots,-\sigma_1+h)$ and $\delta_m=(m-1,m-2,\ldots,0)$, so the result follows. □

Observe that in this result, λ is not necessarily a partition, but it could be an arbitrary integer m-tuple. In such a case, $s_\lambda(x)$ is still well defined by $|x^{\lambda+\delta_m}|/|x^{\delta_m}|$.

Lemma A.5. *Suppose $|x|=m$, $|y|=n$, and h, p and q positive integers with $m=p+q$. Let $\kappa=(\kappa_1,\ldots,\kappa_q)$, $\eta=(\eta_1,\eta_2,\ldots)$, and μ be partitions, and $\nu=(\kappa_1,\ldots,\kappa_q,\eta_1,\eta_2,\ldots)$. Then,*

$$\sum_{x'+x''}\frac{\left(\prod x'\right)^q\left(\prod x''\right)^h s_{\overline{\eta};\mu}(x'/y)s_{\kappa+(h^q)}(\overline{x}''/\overline{y})}{E(x',x'')}=s_{\overline{\nu};\mu}(x/y) \qquad \text{(A.21)}$$

where the sum is over all possible decompositions $x=x'+x''$ with the size of x' equal to p and the size of x'' equal to q.

Proof. We shall use induction on the number n of variables y. Suppose $n=0$. Since, $s_{\overline{\eta};\mu}(x)=\left(\prod_i x_i\right)^{-\eta_1}s_\xi(x)$ with $\xi=(\mu_1+\eta_1,\mu_2+\eta_1,\ldots,\eta_1-\eta_2,0)$ [47], this lemma coincides with lemma A.4 with $g=q-\eta_1$, $\varphi=\xi$ and $\sigma=\kappa+(h^p)$. Suppose that $n>0$ and denote by $y^{(n)}=(y_1,\ldots,y_n)$. We can use lemma A.2 to isolate y_n, giving

$$\sum_{x'+x''}\frac{\left(\prod x'\right)^q\left(\prod x''\right)^h s_{\overline{\eta};\mu}(x'/y)s_{\kappa+(h^q)}(\overline{x}''/\overline{y})}{E(x',x'')}$$

$$=\sum_{x'+x''}\frac{\left(\prod x'\right)^q\left(\prod x''\right)^h}{E(x',x'')}\left(\sum_{\alpha,\beta}s_{\overline{\beta};\alpha}(x'/y^{(n-1)})y_n^{a-b}\right)\left(\sum_\gamma s_{\gamma+(h^q)}(\overline{x}''/\overline{y}^{(n-1)})y_n^{-c}\right)$$

where μ/α, η/β and κ/γ are vertical strips of length a, b and c, respectively. Rearranging terms, this sum equals

$$\sum_{\alpha,\beta,\gamma}\left(\sum_{x'+x''}\frac{\left(\prod x'\right)^q\left(\prod x''\right)^h}{E(x',x'')}s_{\overline{\beta};\alpha}(x'/y^{(n-1)})s_{\gamma+(h^q)}(\overline{x}''/\overline{y}^{(n-1)})\right)y_n^{a-b-c}.$$

Let $\tau=(\gamma_1,\ldots,\gamma_q,\beta_1,\beta_2,\ldots)$. Using induction, the sum reduces to $\sum_{\alpha,\tau}s_{\overline{\tau};\alpha}(x/y^{(n-1)})$ $y_n^{a-(b+c)}$ where μ/α is a vertical strip of length a and $(\nu-\tau)_i\in\{0,1\}$ with $|\nu-\tau|=b+c$. Using lemma A.3, this is equal to $s_{\overline{\nu};\mu}(x/y)$. □

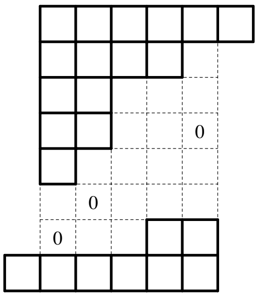


Figure 2. $\overline{\nu};\mu = (\overline{2},\overline{6});(6,4,2,2,1)$.

Now we are in a position to prove the main theorem, first for a special case (theorem A.6), and using this finally the general case (theorem A.8). The special case consists of a subclass of all standard and critical composite partitions without overlap. This subclass is characterized by $n = l + a - 1$; in other words, we will consider standard composite partitions where the first zero in the atypicality matrix (the zero in the row with index k) is *in the last column*. An example is given in figure 2, with $\overline{\nu};\mu = (\overline{2},\overline{6});(6,4,2,2,1)$ in $\mathfrak{gl}(8|5)$. In this case, $k = 4, l = 3, a = 3$ and $r = -2$.

Theorem A.6. *Suppose* $\overline{\nu};\mu$ *is a standard and critical composite partition with no overlap in* $\mathfrak{gl}(m|n)$ *with* $n = l + a - 1$. *Then,*

$$\mathrm{ch}(V_{\overline{\nu};\mu}) = \pm s_{\overline{\nu};\mu}(x/y).$$

Proof. Let $p = k + a - 1$ and $q = m - k - a + 1$. Note that in this special case $r = -q$. First substitute y_j by $-y_j$ in the determinantal formula (corollary 4.2) of $\mathrm{ch}(V_{\overline{\nu};\mu})$. Next, take the Laplace expansion of this determinant with respect to columns $1, 2, \ldots, n + k - 1$. So, with the elements of x' denoted by $x_{r_1}, \ldots, x_{r_p}$ and those of x'' by $x_{s_1}, \ldots, x_{s_q}$, we have the following expression for the character:

$$\frac{E(x,y)}{D(x)D(y)}\det\begin{pmatrix} 0 & \left(y_j^{\mu'_i+n-m-i}\right)_{\substack{i=1,\ldots,n-a\\ j=1,\ldots,n}} & 0 \\ \left(x_i^{\mu_j+m-n-j}\right)_{\substack{i=1,\ldots,m\\ j=1,\ldots,k-1}} & \left(\frac{y_j^r}{x_i^r(x_i-y_j)}\right)_{\substack{i=1,\ldots,m\\ j=1,\ldots,n}} & \left(x_i^{-\nu'_{m-j+1}+m-j}\right)_{\substack{i=1,\ldots,m\\ j=k+a,\ldots,m}} \end{pmatrix}$$

$$= \frac{E(x,y)}{D(x)D(y)}\sum_{x'+x''}(-1)^P \begin{vmatrix} 0 & \left(y_j^{\mu'_i+n-m-i}\right)_{\substack{i=1,\ldots,n-a\\ j=1,\ldots,n}} \\ \left(x_{r_i}^{\mu_j+m-n-j}\right)_{\substack{i=1,\ldots,p\\ j=1,\ldots,k-1}} & \left(\frac{y_j^r}{x_{r_i}^r(x_{r_i}-y_j)}\right)_{\substack{i=1,\ldots,n-a\\ j=1,\ldots,k-1}} \end{vmatrix} \left|\left(x_{s_i}^{-\nu'_j+j-1}\right)_{\substack{i=1,\ldots,q\\ j=1,\ldots,q}}\right|,$$

with $P = \frac{(n+k-1)(n+k)}{2} + r_1 + \cdots + r_p$.

In the right-hand side of this expression we can rewrite the first determinant, using [38, theorem 3.4]

$$\frac{E(x',y)}{D(x')D(y)}\begin{vmatrix} 0 & \left(y_j^{\mu'_i+n-m-i}\right)_{\substack{i=1,\ldots,n-a\\ j=1,\ldots,n}} \\ \left(x_{r_i}^{\mu_j+m-n-j}\right)_{\substack{i=1,\ldots,p\\ j=1,\ldots,k-1}} & \left(\frac{y_j^r}{x_{r_i}^r(x_{r_i}-y_j)}\right)_{\substack{i=1,\ldots,p\\ j=1,\ldots,n}} \end{vmatrix}$$

$$= \frac{E(x',y)}{D(x')D(y)}\left(\frac{\prod_{i=1}^{p}x_{r_i}}{\prod_{j=1}^{n}y_j}\right)^{q}\begin{vmatrix} 0 & \left(y_j^{\mu_i'+n-p-i}\right)_{\substack{i=1,\dots,n-a\\ j=1,\dots,n}} \\ \left(x_{r_i}^{\mu_j+p-n-j}\right)_{\substack{i=1,\dots,p\\ j=1,\dots,k-1}} & \left(\frac{1}{x_{r_i}-y_j}\right)_{\substack{i=1,\dots,p\\ j=1,\dots,n}}\end{vmatrix}$$

$$= \pm\left(\frac{\prod_{i=1}^{p}x_{r_i}}{\prod_{j=1}^{n}y_j}\right)^{q} s_\mu(x'/-y),$$

where the minus sign depends on the partition μ only. So, the Laplace expansion equals

$$\pm\sum_{x'+x''}(-1)^P\frac{E(x,y)}{D(x)D(y)}\frac{D(x')D(y)}{E(x',y)}\left(\frac{\prod_{i=1}^{p}x_{r_i}}{\prod_{j=1}^{n}y_j}\right)^{q} s_\mu(x'/-y)D(x'')s_\nu(\overline{x}'')$$

$$= \pm\sum_{x'+x''}\frac{E(x'',y)}{E(x',x'')}\left(\frac{\prod_{i=1}^{p}x_{r_i}}{\prod_{j=1}^{n}y_j}\right)^{q} s_\mu(x'/-y)s_\nu(\overline{x}'') \qquad \text{(formula(A.20))}$$

$$= \pm\sum_{x'+x''}\frac{(-1)^{qn}\left(\prod_{i=1}^{q}x_{s_i}^{n}\right)\left(\prod_{j=1}^{n}y_j^{q}\right)E(\overline{x}'',\overline{y})}{E(x',x'')}\frac{\left(\prod_{i=1}^{p}x_{r_i}^{q}\right)}{\left(\prod_{j=1}^{n}y_j^{q}\right)}s_\mu(x'/-y)s_\nu(\overline{x}''). \qquad \text{(formula(A.16))}$$

Applying a special case of the Sergeev–Pragacz formula [38, equation (1.12)] and lemma A.5, the sum equals

$$\pm\sum_{x'+x''}\frac{\left(\prod_{i=1}^{p}x_{r_i}\right)^{q}\left(\prod_{i=1}^{q}x_{s_i}\right)^{n}}{E(x',x'')}s_\mu(x'/-y)s_{\nu+(n^q)}(\overline{x}''/-\overline{y}) = \pm s_{\overline{\nu};\mu}(x/-y).$$

Substituting every y_j by $-y_j$, yields the theorem. □

The next corollary follows immediately from this theorem and the fact that

$$s_{\overline{\nu};\mu}(x/y) = s_{\overline{\nu'};\mu'}(y/x). \qquad \text{(A.22)}$$

Corollary A.7. *Suppose $\overline{\nu};\mu$ is a standard and critical composite partition with no overlap in $\mathfrak{gl}(m|n)$ with $m = k+a-1$. Then,*

$$\mathrm{ch}(V_{\overline{\nu};\mu}) = \pm s_{\overline{\nu};\mu}(x/y).$$

Now we can prove the final theorem.

Theorem A.8. *Let $\overline{\nu};\mu$ be a standard and critical composite partition with no overlap in $\mathfrak{gl}(m|n)$. Then,*

$$\mathrm{ch}(V_{\overline{\nu};\mu}) = \pm s_{\overline{\nu};\mu}(x/y).$$

Proof. Let $p = l+a-1$ and $q = n-l-a+1$. First substitute y_j by $-y_j$ in the determinantal formula of $\mathrm{ch}(V_{\overline{\nu};\mu})$. Next, take the Laplace expansion of the determinant with respect to rows $1, 2, \dots, m+l-1$. So, with the elements of y' denoted by $y_{r_1}, \dots, y_{r_p}$ and those of y'' by $y_{s_1}, \dots, y_{s_q}$, we have the following expression:

$$\frac{E(x,-y)}{D(x)D(y)}\det\begin{pmatrix} 0 & \left(y_j^{\mu_i'+n-m-i}\right)_{\substack{i=1,\dots,l-1\\ j=1,\dots,n}} & 0 \\ \left(x_i^{\mu_j+m-n-j}\right)_{\substack{i=1,\dots,m\\ j=1,\dots,k-1}} & \left(\frac{y_j^r}{x_i^r(x_i-y_j)}\right)_{\substack{i=1,\dots,m\\ j=1,\dots,n}} & \left(x_i^{-\nu'_{m-j+1}+m-j}\right)_{\substack{i=1,\dots,m\\ j=k+a,\dots,m}} \\ 0 & \left(y_j^{-\nu'_{n-i+1}+n-i}\right)_{\substack{i=l+a,\dots,n\\ j=1,\dots,n}} & 0\end{pmatrix}$$

$$= \frac{E(x,-y)}{D(x)D(y)}\sum_{x'+x''}(-1)^P C_1\left|\left(y_{s_j}^{-\nu'_{q-i+1}+q-i}\right)_{\substack{i=1,\dots,q\\ j=1,\dots,q}}\right|,$$

with $P = \frac{(m+l-1)(m+l)}{2} + r_1 + \cdots + r_p$. The determinant C_1 equals

$$C_1 = \begin{vmatrix} 0 & \left(y_{r_j}^{\mu_i'+n-m-i}\right)_{\substack{i=1,\dots,l-1\\ j=1,\dots,p}} & 0 \\ \left(x_i^{\mu_j+m-n-j}\right)_{\substack{i=1,\dots,m\\ j=1,\dots,k-1}} & \left(\frac{y_{r_j}^r}{x_i^r(x_i-y_{r_j})}\right)_{\substack{i=1,\dots,m\\ j=1,\dots,p}} & \left(x_i^{-\nu_{m-j+1}+m-j}\right)_{\substack{i=1,\dots,m\\ j=k+a,\dots,m}} \end{vmatrix}$$

$$= \left(\frac{\prod_{j=1}^p y_{r_j}}{\prod_{i=1}^m x_i}\right)^q \begin{vmatrix} 0 & \left(y_{r_j}^{\mu_i'+p-m-i}\right)_{i,j} & 0 \\ \left(x_i^{\mu_j+m-p-j}\right)_{i,j} & \left(\frac{y_{r_j}^{r'}}{x_i^{r'}(x_i-y_{r_j})}\right)_{i,j} & \left(x_i^{-\nu_{m-j+1}+q+m-j}\right)_{i,j} \end{vmatrix},$$

where $r' = r - q = p - m + k - l$. Thus, this determinantal expression coincides with the determinantal formula (4.7) of $\mathrm{ch}(V_{\overline{\eta};\mu})$ in $\mathfrak{gl}(m|p)$ with $\eta = (\nu_1, \dots, \nu_{m-k-a+1}) - (q^{m-k-a+1})$. According to theorem A.6, the determinant C_1 equals

$$\pm \frac{D(x)D(y')}{E(x, y')} \left(\frac{\prod_{j=1}^p y_{r_j}}{\prod_{i=1}^m x_i}\right)^q s_{\overline{\eta};\mu}(x/-y').$$

So, $\eta' = (\nu_{q+1}', \nu_{q+2}', \dots)$ and let $\beta' = (\nu_1', \dots, \nu_q')$. As the minus sign depends on the composite partition $\overline{\eta}; \mu$ only, the Laplace expansion equals

$$\pm \sum_{y'+y''} (-1)^P \frac{E(x, y)}{D(x)D(y)} \frac{D(x)D(y')}{E(x, y')} \left(\frac{\prod_{j=1}^p y_{r_j}}{\prod_{i=1}^m x_i}\right)^q s_{\overline{\eta};\mu}(x/-y') D(y'') s_{\beta'}(\overline{y}'')$$

$$= \pm \sum_{y'+y''} \frac{E(x, y'')}{E(y', y'')} \left(\frac{\prod_{j=1}^p y_{r_j}}{\prod_{i=1}^m x_i}\right)^q s_{\overline{\eta};\mu}(x/-y') s_{\beta'}(\overline{y}'')$$

$$= \pm \sum_{y'+y''} \frac{\left(\prod_{i=1}^m x_i\right)^q \left(\prod_{j=1}^q y_{s_j}\right)^m}{E(y', y'')} \left(\frac{\prod_{j=1}^p y_{r_j}}{\prod_{i=1}^m x_i}\right)^q s_{\overline{\eta};\mu}(x/-y') s_{\beta'}(\overline{y}'') E(\overline{y}'', \overline{x}),$$

where we have used (A.16). Simplifying and applying again the special case of the Sergeev–Pragacz formula and (A.22), this becomes

$$\pm \sum_{y'+y''} \frac{\left(\prod_{j=1}^p y_{r_j}\right)^q \left(\prod_{j=1}^q y_{s_j}\right)^m}{E(y', y'')} s_{\overline{\eta'};\mu'}(-y'/x) s_{\beta'+(m^q)}(-\overline{y}''/\overline{x}) = \pm s_{\overline{\nu}';\mu'}(-y/x).$$

Applying (A.22) and substituting every y_j by $-y_j$, leads to the final result. □

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